OCR
Oxford Cambridge and RSA

Wednesday 25 May 2022

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Worked Solutions

Printed Answer Booklet

Time allowed: 2 hours 40 minutes

You must have:
- Question Paper Y420/01 (inside this document)
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

R red level
- longer questions (6+ marks)
- higher level problem solving
- harder A Level content

A amber level
- Shorter questions (3 - 6 marks)
- low level problem solving
- harder AS/easier A Level content.

G green level
- Short questions (1 - 3 marks)
- minimal problem solving
- AS/easier A Level content.

E explanation

Section A (37 marks)

Q1: Series (as) G
Q2: Calculus (a level) A
Q3: Hyperbolic Functions (a level) G
Q4: Matrices (a level) G A
Q5: Polar Coordinates (a level) G
Q6: Proof (as) G

Section B (107 marks)

Q7: Calculus (a level) R
Q8: Complex Numbers (as) A R
Q9: Series (a level) G A E
Q10: Algebra (as) G A
Q11: Complex Numbers (a level) G A
Q12: Differential Equations (a level) R
Q13: Vectors (a level) G A
Q14: Complex Numbers (a level) G R
Q15: Differential Equations (a level) G A R

Grade Boundaries

Grade	A*	A	B	C	D	E	U
Mark / 144	109	85	67	49	31	13	0
Scaled/ 180	136	106	84	61	39	16	0

x 1.25

note: the scaled score is added to the scores in the other modules to find an overall grade, not the raw mark

Turn over

Section A (37 marks)

G

1 (a) By considering $(r+1)^3 - r^3$, find $\sum_{r=1}^{n}(3r^2+3r+1)$. **[3]**

First let's expand out $(r+1)^3 - r^3$

$(r+1)^3 - r^3$

$= r^3 + 3r^2 + 3r + 1 - r^3$

$= 3r^2 + 3r + 1$ ← this what we are summing.

Now we use this to find the sum.

$\sum_{r=1}^{n}(3r^2+3r+1) = \sum_{r=1}^{n}(r+1)^3 - r^3$

$= (\cancel{2^3} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + (\cancel{4^3} - \cancel{3^3})$

$+ \dots$

$+ (\cancel{n^3} - \cancel{(n-1)^3}) + ((n+1)^3 - \cancel{n^3})$

Method of Differences

$= (n+1)^3 - 1 = n^3 + 3n^2 + 3n + 1 - 1$

$= n^3 + 3n^2 + 3n = n(n^2+3n+3)$

G

(b) Use this result to find $\sum_{r=1}^{n} r(r+1)$, expressing your answer in fully factorised form. **[4]**

We should first need to find $\Sigma r(r+1)$ in our sum above.

$\sum_{r=1}^{n}(3r^2+3r+1) = 3\sum_{r=1}^{n}(r(r+1)) + \sum_{r=1}^{n} 1$

so $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}\left[\sum_{r=1}^{n}(3r^2+3r+1) - \sum_{r=1}^{n} 1\right]$

Now we can sub in our sum from a).

using $\sum_{r=1}^{n} 1 = n$

$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}\left[n(n^2+3n+3) - n\right]$

$= \frac{1}{3}n(n^2+3n+3-1)$

$= \frac{1}{3}n(n^2+3n+2)$

$= \frac{1}{3}n(n+1)(n+2)$

always check of quadratics will factorise

hence $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

A

2 In this question you must show detailed reasoning.

Find the exact value of $\int_3^{\infty} \frac{1}{x^2-4x+5}\,dx$. **[5]**

This is an improper integral as one of the limits is ∞. So firstly we need to modify the limits of our integral.

$$\int_3^{\infty} \frac{1}{x^2-4x+5}\,dx = \lim_{a\to\infty} \int_3^{a} \frac{1}{x^2-4x+5}\,dx.$$

The quadratic in the denominator does not factorise. So we must complete the square and use a trig / hyperbolic trig substitution.

$$x^2-4x+5$$
$$= (x-2)^2-(2)^2+5$$
$$= (x-2)^2+1$$

So $\lim_{a\to\infty} \int_3^{a} \frac{1}{x^2-4x+5}\,dx = \lim_{a\to\infty} \int_3^{a} \frac{1}{(x-2)^2+1}\,dx$

From the formula book, $\int \frac{1}{x^2+1}\,dx = \arctan x + c$

So $\lim_{a\to\infty} \int_3^{a} \frac{1}{(x-2)^2+1}\,dx = \lim_{a\to\infty} \left[\arctan(x-2)\right]_3^a$

$$= \lim_{a\to\infty} \left(\arctan(a-2) - \arctan 1\right)$$

$$= \lim_{a\to\infty} \left(\arctan(a-2) - \frac{\pi}{4}\right)$$

as $a\to\infty$, $\arctan(a-2) \to \frac{\pi}{2}$ as $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$

so $\int_3^{\infty} \frac{1}{x^2-4x+5}\,dx = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

G

3 In this question you must show detailed reasoning.

Solve the equation $3\cosh x = 2\sinh^2 x$, giving your solutions in exact logarithmic form. **[6]**

First we need to write our equation in terms of $\cosh x$.

From the formula book, $\cosh^2 x - \sinh^2 x = 1$. So

$$\cosh^2 x - \sinh^2 x = 1$$

$$\Rightarrow \sinh^2 x = \cosh^2 x - 1$$

Substituting into our equation

$$3\cosh x = 2(\cosh^2 x - 1)$$

$$3\cosh x = 2\cosh^2 x - 2$$

$$2\cosh^2 x - 3\cosh x - 2 = 0$$

This is a quadratic in $\cosh x$. So

$$(\cosh x - 2)(2\cosh x + 1) = 0$$

$$\cosh x = 2, \quad \cosh x = -\tfrac{1}{2}$$

$$\cosh x = -\tfrac{1}{2}$$

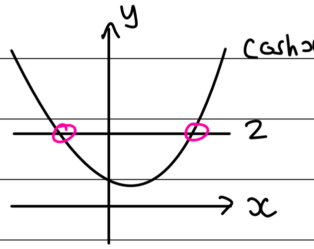
no $x \in \mathbb{R}$ as $\cosh x \geq 0$

$$\cosh x = 2$$

$$x = \operatorname{arcosh} 2 = \ln\left(2 + \sqrt{2^2 - 1}\right)$$

$$= \ln(2 + \sqrt{3})$$

By considering the graph, we also need the negated value of the root above.



So $x = -\ln(2 + \sqrt{3})$ as well.

hence $x = \ln(2 + \sqrt{3})$, $x = -\ln(2 + \sqrt{3})$

G

4 **(a)** A transformation with associated matrix $\begin{pmatrix} m & 2 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & 3 \end{pmatrix}$, where m is a constant, maps the vertices of a cube to points that all lie in a plane.

Find m. **[3]**

If the transformation maps all points to a plane, the determinant of the matrix is 0.

$$\begin{vmatrix} m & 2 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & 3 \end{vmatrix} = m\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}$$

$$= m(3) - 2(4) + 1(-2)$$

$$= 3m - 10$$

So $3m - 10 = 0 \Rightarrow m = \frac{10}{3}$

A

(b) The transformations S and T of the plane have associated matrices **M** and **N** respectively, where $\mathbf{M} = \begin{pmatrix} k & 1 \\ -3 & 4 \end{pmatrix}$ and the determinant of **N** is $3k + 1$. The transformation U is equivalent to the combined transformation consisting of S followed by T.

Given that U preserves orientation and has an area scale factor 2, find the possible values of k. **[4]**

The determinant of the matrix representing U is 2.

So $\det NM = 2$

NM represents the combined transformation M then N.

Also $\det NM = \det N \times \det M$

$\det N = 3k + 1$

$\det M = 4k - (-3) = 4k + 3$

So $(3k+1)(4k+3) = 2$

$12k^2 + 9k + 4k + 3 = 2$

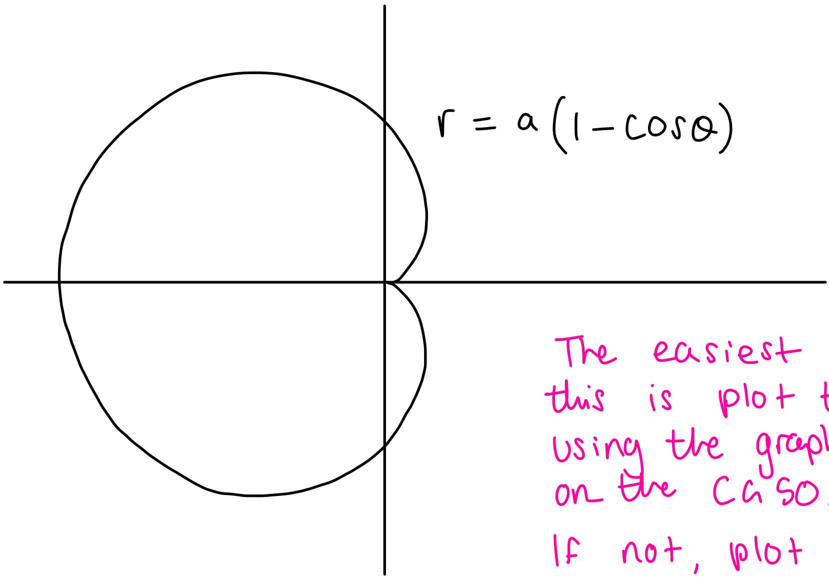
$12k^2 + 13k + 1 = 0$

$(12k + 1)(k + 1) = 0$

So $k = -\frac{1}{12}$ or $k = -1$

G1

5 (a) Sketch the polar curve $r = a(1-\cos\theta)$, $0 \leqslant \theta < 2\pi$, where a is a positive constant. **[2]**



The easiest way to do this is plot the graphs using the graph function on the CASIO.
If not, plot a few points and use this to draw the curve.

G1

(b) Determine the exact area of the region enclosed by the curve. **[5]**

From looking at the graph above, the bounds for the area are 0 and 2π.

also in formula book the area $= \frac{1}{2}\int r^2 \, d\theta$

so area $= \frac{1}{2}\int_0^{2\pi} (a(1-\cos\theta))\, d\theta = \frac{1}{2}\int_0^{2\pi} a^2(1-\cos\theta)^2 \, d\theta$

$= \frac{a^2}{2}\int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta \, d\theta$

using $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$= \frac{a^2}{2}\int_0^{2\pi} 1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\, d\theta$

$= \frac{a^2}{2}\int_0^{2\pi} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta \, d\theta$

$\int \cos 4\theta \, d\theta = \frac{1}{4}\sin 4\theta + c$

$= \frac{a^2}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$

$= \frac{a^2}{2}\left(\frac{3}{2}(2\pi) - 2\sin 2\pi + \frac{1}{4}\sin 4\pi - \left(\frac{3}{2}(0) - 2\sin 0 + \frac{1}{4}\sin 0\right)\right)$

$= \frac{a^2}{2}\left(3\pi - 0 + \frac{1}{4}(0) - \left(0 - 2(0) + \frac{1}{4}(0)\right)\right)$

$= \frac{a^2}{2} \times 3\pi = \frac{3}{2}a^2\pi$ units^2

G

6 Prove by mathematical induction that $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 \\ 1-2^n & 1 \end{pmatrix}$ for all positive integers n. **[5]**

Step one: base case

When $n = 1$, $LHS = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^1 = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$

$RHS = \begin{pmatrix} 2^1 & 0 \\ 1-2^1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$ $\therefore$ true for $n=1$.

Step two: assumption

Assume true for $n=k$, so $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 0 \\ 1-2^k & 1 \end{pmatrix}$

Step three: inductive step

Using the assumed result for $n=k$,

$$\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^k \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^k & 0 \\ 1-2^k & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times 2^k & 0 \\ 2(1-2^k)-1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{(k+1)} & 0 \\ 1-2^{(k+1)} & 1 \end{pmatrix}$$ $\therefore$ true for $n = k+1$.

Step four: conclusion

If the result is true for $n=k$, it is true for $n=k+1$. Since it is true for $n=1$, it is true for all positive integer values of n.

Section B (107 marks)

R

7 **In this question you must show detailed reasoning.**

Show that $\int_2^3 \frac{x+1}{(x-1)(x^2+1)}\,dx = \frac{1}{2}\ln 2.$ **[9]**

We can find the integral using partial fractions.

$$\frac{x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + (Bx+C)(x-1)}{(x-1)(x^2+1)}$$

hence $A(x^2+1) + (Bx+C)(x-1) = x+1$

$Ax^2 + A + Bx^2 - Bx + Cx - C = x + 1$

Find A, B and C

$x^2: \quad A + B = 0$

$x: \quad -B + C = 1$

constants: $\quad A - C = 1$

→ solving simultaneously gives $A = 1,\ B = -1,\ C = 0$

so $\int_2^3 \frac{x+1}{(x-1)(x^2+1)}\,dx = \int_2^3 \frac{1}{x-1} - \frac{x}{x^2+1}\,dx$

$$= \left[\ln|x-1| - \tfrac{1}{2}\ln|x^2+1|\right]_2^3$$

using $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$

$$= \ln 2 - \tfrac{1}{2}\ln 10 - \left(\ln 1 - \tfrac{1}{2}\ln 5\right)$$

$$= \ln 2 - \tfrac{1}{2}\ln 10 + \ln 1 + \tfrac{1}{2}\ln 5$$

$$= \ln 2 + \tfrac{1}{2}(\ln 5 - \ln 10)$$

$$= \ln 2 + \tfrac{1}{2}\ln\tfrac{5}{10}$$

$$= \ln 2 - \tfrac{1}{2}\ln 2 = \tfrac{1}{2}\ln 2$$ as required

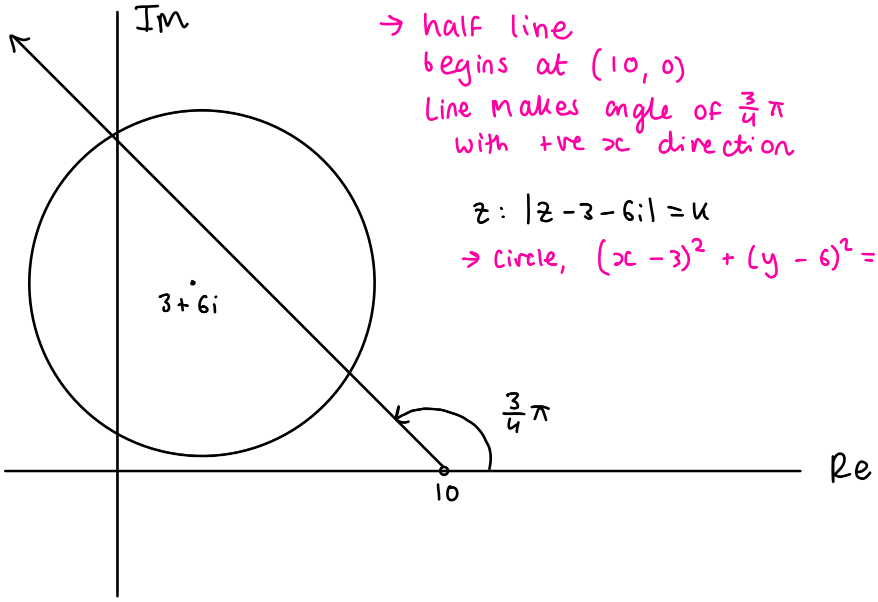
simplify into required result

(answer space continued on next page)

8 Two sets of complex numbers are given by $\left\{z : \arg(z-10) = \frac{3}{4}\pi\right\}$ and $\{z : |z-3-6\mathrm{i}| = k\}$, where k is a positive constant. In an Argand diagram, one of the points of intersection of the two loci representing these sets lies on the imaginary axis.

A

(a) Sketch the loci on an Argand diagram. **[4]**



$z : \arg(z-10) = \frac{3}{4}\pi$

→ half line
begins at (10, 0)
Line makes angle of $\frac{3}{4}\pi$
with +ve x direction

$z : |z-3-6i| = k$

→ Circle, $(x-3)^2 + (y-6)^2 = k^2$

R

(b) In this question you must show detailed reasoning.

Find the complex numbers represented by the points of intersection. **[7]**

This is essentially a coordinate geometry question.
So first we re-write our loci in terms of x and y.

$\arg(z-10) = \frac{3}{4}\pi$ is a line, gradient $\tan\left(\frac{3}{4}\pi\right) = -1$,
passing through $(10, 0)$. So

$$y - 0 = -1(x-10)$$

$$y = -x + 10$$

$|z-3-6i| = k$ is a circle, centre $(3, 6)$, radius k.

so $(x-3)^2 + (y-6)^2 = k^2$

We know they intersect on the imaginary axis,
so intersect at $x = 0$. When $x = 0$, $y = -0 + 10 = 10$

So $(0-3)^2 + (10-6)^2 = k^2$

$k^2 = 25 \Rightarrow k = 5$

8 b) continued

so now we solve for the other point of intersection.

subbing $y = -x + 10$ into $(x-3)^2 + (y-6)^2 = 25$,

$$(x-3)^2 + (-x+10-6)^2 = 25$$

$$x^2 - 6x + 9 + x^2 - 8x + 16 = 25$$

$$2x^2 - 14x = 0$$

$$2x(x-7) = 0$$

$$x = 0, \quad x = 7$$

$$y = 10, \quad y = 3$$

so the points of intersection are represented $10i$ and $7 + 3i$.

A

9 The function $f(x)$ is defined by $f(x) = \ln(1 + \sinh x)$.

E

(a) Given that k lies in the domain of this function, explain why k must be greater than $\ln(\sqrt{2} - 1)$. **[2]**

$\ln(x)$ is only valid for $x > 0$.

so $1 + \sinh x > 0$

$\sinh x > -1$

$x > \text{arsinh}(-1)$

$x > \ln\left(-1 + \sqrt{(-1)^2 + 1}\right)$ using $\text{arsinh}\, x = \ln(x + \sqrt{x^2 + 1})$

$x > \ln(\sqrt{2} - 1)$

hence k must be greater than $\ln(\sqrt{2} - 1)$.

A

(b) (i) Find $f'(x)$. **[2]**

$f(x) = \ln(1 + \sinh x)$

by chain rule,

$f(x) = \ln u$ $\quad f'(x) = \frac{1}{u}$

$u = 1 + \sinh x$ $\quad u' = \cosh x$

so $f'(x) = \frac{1}{1 + \sinh x} \times \cosh x$

$= \frac{\cosh x}{1 + \sinh x}$

A

(ii) Show that $f''(x) = \frac{a \sinh x + b}{(1 + \sinh x)^2}$, where a and b are integers to be determined. **[3]**

by quotient rule,

$u = \cosh x$ $\quad u' = \sinh x$

$v = 1 + \sinh x$ $\quad v' = \cosh x$

so $f''(x) = \frac{\sinh x(1 + \sinh x) - \cosh^2 x}{(1 + \sinh x)^2}$ using $\cosh^2 x - \sinh^2 x = 1$

$= \frac{\sinh x + \sinh^2 x - \cosh^2 x}{(1 + \sinh x)^2} = \frac{\sinh x - 1}{(1 + \sinh x)^2}$

$(a = 1,\ b = -1)$

G1

(c) Hence find a quadratic approximation to $f(x)$ for small values of x. **[3]**

As given in the formula book, the general Maclaurin series is $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$

$f(0) = \ln(1 + \sinh 0) = \ln 1 = 0$

$f'(0) = \frac{\cosh 0}{1+\sinh 0} = \frac{1}{1} = 1$

$f''(0) = \frac{\sinh(0) - 1}{(1+\sinh(0))^2} = \frac{-1}{1} = -1$

so $f(x) = 0 + 1(x) + \frac{-1}{2}(x^2) + \ldots$

$f(x) = x - \frac{1}{2}x^2 + \ldots$

hence a quadratic approximation for $f(x)$ is $x - \frac{1}{2}x^2$.

G1

(d) Find the percentage error in this approximation when $x = 0.1$. **[2]**

$f(0.1) = \ln(1 + \sinh 0.1) = 0.0954\ldots$

$f(0.1) \approx 0.1 - \frac{1}{2}(0.1)^2 = 0.095$

$\%\ \text{error} = \frac{\text{true value} - \text{approximation}}{\text{true value}} \times 100$

$= \frac{0.0954\ldots - 0.095}{0.0954\ldots} \times 100$

$= 0.4837\ldots = 0.484\,\%$ (3sf)

10 The equation

$$4x^4 + 16x^3 + ax^2 + bx + 6 = 0,$$

where a and b are real, has roots α, $\frac{2}{\alpha}$, β and 3β.

A

(a) Given that $\beta < 0$, determine all 4 roots. **[6]**

Using $\sum \alpha\beta\gamma\delta = \frac{e}{a}$ (product of roots)

$\alpha \times \frac{2}{\alpha} \times \beta \times 3\beta = \frac{6}{4}$

$6\beta^2 = \frac{6}{4}$

$\beta^2 = \frac{1}{4}$

$\beta = \pm\frac{1}{2} \Rightarrow \beta < 0$, so $\beta = -\frac{1}{2}$

Using $\sum \alpha = -\frac{b}{a}$ (sum of roots)

$\alpha + \frac{2}{\alpha} - \frac{1}{2} - \frac{3}{2} = -\frac{16}{4}$

$\alpha + \frac{2}{\alpha} + 2 = 0$

$\alpha^2 + 2\alpha + 1 = 0$

$\alpha = \frac{-2 \pm \sqrt{(2)^2 - 4 \times 1 \times 1}}{2(1)} = -1 \pm i$ $\qquad \frac{2}{-1 \pm i} = -1 \mp i$

So roots are $-\frac{1}{2}$, $-\frac{3}{2}$, $-1+i$, $-1-i$

G

(b) Determine the values of a and b. **[4]**

Using $\sum \alpha\beta = \frac{c}{a}$ (sum of product of 2 roots)

$\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) + \left(-\frac{1}{2}\right)(-1+i) + \left(-\frac{1}{2}\right)(-1-i)$

$+ \left(-\frac{3}{2}\right)(-1+i) + \left(-\frac{3}{2}\right)(-1-i)$

$+ (-1+i)(-1-i) = \frac{a}{4}$

$\Rightarrow \frac{a}{4} = \frac{27}{4} \Rightarrow a = 27$

Using $\sum \alpha\beta\gamma = -\frac{d}{a}$ (sum of product of 3 roots)

$\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-1+i) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-1-i)$

$+ \left(-\frac{1}{2}\right)(-1+i)(-1-i) + \left(-\frac{3}{2}\right)(-1+i)(-1-i)$

$= -\frac{6}{4}$

$\Rightarrow -\frac{6}{4} = -\frac{11}{2}$

$b = 22$

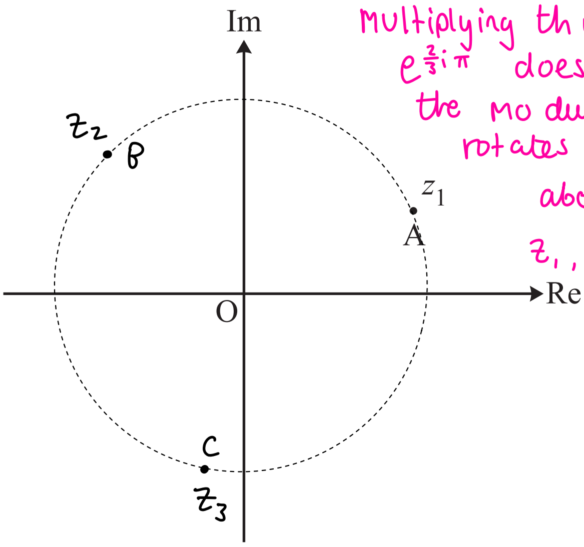
$a =$ 27

$b =$ 22

The complex numbers z_2 and z_3 are $z_1 e^{\frac{2}{3}i\pi}$ and $z_1 e^{\frac{4}{3}i\pi}$ respectively.

G

(a) **(i)** On the copy of the Argand diagram in the Printed Answer Booklet, mark the points B and C representing the complex numbers z_2 and z_3. **[2]**



Multiplying through by $e^{\frac{2}{3}i\pi}$ does not change the modulus of z_1, but rotates it by $\frac{2}{3}\pi$ about the origin.

z_1, z_2 and z_3 will form an equilateral triangle with the origin as the centre.

A

(ii) Show that $z_1 + z_2 + z_3 = 0$. **[2]**

$z_1 + z_2 + z_3 = z_1 + z_1 e^{\frac{2}{3}i\pi} + z_1 e^{\frac{4}{3}i\pi}$

This forms a geometric series, $a = z_1$, $r = e^{\frac{2}{3}i\pi}$

So $z_1 + z_2 + z_3 = \dfrac{z_1\left(1 - (e^{\frac{2}{3}i\pi})^3\right)}{1 - e^{\frac{2}{3}i\pi}} = \dfrac{z_1(1 - e^{2i\pi})}{1 - e^{\frac{2}{3}i\pi}}$

$e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1 + i(0) = 1$

so $z_1 + z_2 + z_3 = \dfrac{z_1(1-1)}{1 - e^{\frac{2}{3}i\pi}} = 0$

$\Rightarrow z_1 + z_2 + z_3 = 0$

A

(b) Given now that z_1, z_2 and z_3 are roots of the equation $z^3 = 8i$, find these three roots, giving your answers in the form $a + ib$, where a and b are real and exact. **[4]**

First we write $8i$ in mod-arg form.

$|8i| = 8$, $\arg 8i = \frac{\pi}{2}$, so $z^3 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

Hence by de Moivre's, $z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$

arguments have $\frac{2}{3}\pi$ added each time

$z_2 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$

$z_3 = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2i$

so roots are $\sqrt{3} + i$, $-\sqrt{3} + i$, $-2i$

R

12 Solve the differential equation $(4-x^2)\frac{dy}{dx} - xy = 1$, given that $y = 1$ when $x = 0$, giving your answer in the form $y = f(x)$. **[9]**

First we need to re-write our DE into a linear 1st order general form DE.

$$(4-x^2)\frac{dy}{dx} - xy = 1$$

$\div (4-x^2)$: $\frac{dy}{dx} - \frac{x}{4-x^2}y = \frac{1}{4-x^2}$

We can solve this by using an integrating factor.

$$I(x) = e^{\int \frac{x}{4-x^2}dx} = e^{\frac{1}{2}\ln|4-x^2|} = (4-x^2)^{\frac{1}{2}}$$

$\int \frac{F'(x)}{F(x)}dx = \ln|F(x)| + c$ $\qquad e^{\ln(F(x))} = f(x)$

Now we multiply through by $I(x)$.

$\times (4-x^2)^{\frac{1}{2}}$: $(4-x^2)^{\frac{1}{2}}\frac{dy}{dx} - x(4-x^2)^{-\frac{1}{2}}y = (4-x^2)^{-\frac{1}{2}}$

The LHS is implicit product rule. $u = y$ $\quad u' = \frac{dy}{dx}$

$v = (4-x^2)^{\frac{1}{2}}$ $\quad v' = -x(4-x^2)^{-\frac{1}{2}}$

$\Rightarrow \frac{d}{dx}\left(y(4-x^2)^{\frac{1}{2}}\right) = (4-x^2)^{\frac{1}{2}}\frac{dy}{dx} - x(4-x^2)^{-\frac{1}{2}}y$

So $\frac{d}{dx}\left(y(4-x^2)^{\frac{1}{2}}\right) = (4-x^2)^{-\frac{1}{2}}$

$$y(4-x^2)^{\frac{1}{2}} = \int \frac{1}{\sqrt{4-x^2}}dx$$

using $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + c$

$$y(4-x^2)^{\frac{1}{2}} = \arcsin\left(\frac{x}{2}\right) + c$$

when $y = 1$, $x = 0$: $1(4)^{\frac{1}{2}} = \arcsin 0 + c \Rightarrow c = 2$

so $y = \dfrac{\arcsin\left(\frac{x}{2}\right) + 2}{\sqrt{4-x^2}}$

13 The points A and B have coordinates $(4, 0, -1)$ and $(10, 4, -3)$ respectively. The planes Π_1 and Π_2 have equations $x-2y=5$ and $2x+3y-z=-4$ respectively.

A

(a) Find the acute angle between the line AB and the plane Π_1. **[4]**

We start by finding the direction vector $\vec{AB}$.

$$\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix}10\\4\\-3\end{pmatrix} - \begin{pmatrix}4\\0\\-1\end{pmatrix} = \begin{pmatrix}6\\4\\-2\end{pmatrix}$$

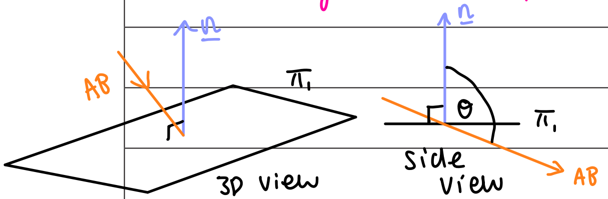
So we can start by find the angle between this direction vector and the normal to π_1 using

$$\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$$

$$\cos\theta = \frac{6(1)+4(-2)+0(-2)}{\sqrt{6^2+4^2+2^2}\sqrt{1^2+2^2}} = -\frac{\sqrt{70}}{70}$$

$$\text{so } \theta = \arccos\left(-\frac{\sqrt{70}}{70}\right) = 96.864\ldots$$

Diagrams help us visualise the angle we have found.



so the angle between the plane and line is $96.864\ldots - 90$

$= 6.864\ldots = 6.86°$

A

(b) Show that the line AB meets Π_1 and Π_2 at the same point, whose coordinates should be specified. **[5]**

So the line has equation $\underline{r} = \begin{pmatrix}4\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}6\\4\\-2\end{pmatrix}$

So $x = 4+6\lambda$, $y = 4\lambda$, $z = -1-2\lambda$

subbing into plane equation

First let's use π_1: $4+6\lambda - 2(4\lambda) = 5$

$-2\lambda = 1$

$\lambda = -\frac{1}{2}$

Now π_2: $2(4+6\lambda) + 3(4\lambda) - (-1-2\lambda) = -4$

$8 + 12\lambda + 12\lambda + 1 + 2\lambda = -4$

$26\lambda = -13$

$\lambda = -\frac{1}{2}$

As they intersect when $\lambda = -\frac{1}{2}$ each time AB meets π_1 and π_2 at the same point.

Coordinates = $\left(4+6\left(-\frac{1}{2}\right), 4\left(-\frac{1}{2}\right), -1-2\left(-\frac{1}{2}\right)\right) = (1, -2, 0)$

G

(c) **(i)** Find $(\mathbf{i}-2\mathbf{j})\times(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$. **[1]**

We can find this using a determinant.

$$\begin{vmatrix} \underline{i} & 1 & 2 \\ \underline{j} & -2 & 3 \\ \underline{k} & 0 & -1 \end{vmatrix} = \underline{i}\begin{vmatrix} -2 & 3 \\ 0 & -1 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}$$

$$= \underline{i}(2) - \underline{j}(-1) + \underline{k}(3+2(2))$$

$$= 2\underline{i} + \underline{j} + 7\underline{k}$$

A

(ii) Hence find the acute angle between the planes Π_1 and Π_2. **[3]**

Above we found the vector product between the normals to each plane. We can use this to find the angle between them. In the FB,

$$\underline{a}\times\underline{b} = |\underline{a}||\underline{b}|\sin\theta\ \hat{n} \qquad \hat{n} \text{ is a unit vector}$$

$$|\underline{a}\times\underline{b}| = |\underline{a}||\underline{b}|\sin\theta\ |\hat{n}| \leftarrow \text{so } |\hat{n}| = 1$$

$$\Rightarrow \sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

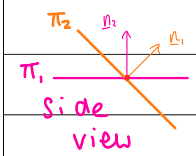
$$\text{so } \sin\theta = \frac{\sqrt{2^2+1^2+7^2}}{\sqrt{1^2+2^2}\sqrt{2^2+3^2+1^2}} = \frac{3\sqrt{105}}{35} \quad \text{(this is acute)}$$

$$\text{so } \theta = \arcsin\left(\frac{3\sqrt{105}}{35}\right) = 61.43\ldots = 61.4^\circ \ (3\text{ sf})$$

A

(iii) Find the shortest distance between the point A and the line of intersection of the planes Π_1 and Π_2. **[4]**

The vector product of two vectors gives a vector perpendicular to the two vectors.



So the vector product in c) i. is the direction vector of the line of intersection. So this line is

$$\underline{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}$$

Recall that the shortest distance between a line $(\underline{r} = \underline{a} + \lambda\underline{d})$ and a point P is given by $\dfrac{|\vec{aP}\times\underline{d}|}{|\underline{d}|}$

iii) continued

$$\overrightarrow{aP} = \underline{p} - \underline{a} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

so $$\overrightarrow{aP} \times \underline{d} = \begin{vmatrix} \underline{i} & 3 & 2 \\ \underline{j} & 2 & 1 \\ \underline{k} & -1 & 7 \end{vmatrix} = \underline{i}\begin{vmatrix} 2 & 1 \\ -1 & 7 \end{vmatrix} - \underline{j}\begin{vmatrix} 3 & 2 \\ -1 & 7 \end{vmatrix} + \underline{k}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= \underline{i}(14+1) - \underline{j}(21+2) + \underline{k}(3-4)$$

$$= 15\underline{i} - 23\underline{j} - \underline{k}$$

$$|\overrightarrow{aP} \times \underline{d}| = \sqrt{15^2+23^2+1^2} = \sqrt{755}$$

$$|\underline{d}| = \sqrt{2^2+1^2+7^2} = 3\sqrt{6}$$

so distance $= \dfrac{\sqrt{755}}{3\sqrt{6}} = 3.7391.... = 3.74$ units (3 sf)

G

14 **(a)** Find $(3-e^{2i\theta})(3-e^{-2i\theta})$ in terms of $\cos 2\theta$. **[2]**

Here we just expand and simplify.

$(3-e^{2i\theta})(3-e^{-2i\theta})$

$= 9 - 3e^{-2i\theta} - 3e^{2i\theta} + 1$

$= 10 - 3(e^{2i\theta} + e^{-2i\theta})$ using $2\cos n\theta = e^{in\theta} + e^{-in\theta}$

$= 10 - 3(2\cos 2\theta)$

$= 10 - 6\cos 2\theta$

R

(b) Hence show that the sum of the infinite series

$$\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{9}\sin 5\theta + \frac{1}{27}\sin 7\theta + \ldots$$

can be expressed as $\dfrac{6\sin\theta}{5-3\cos 2\theta}$. **[6]**

This is a C+iS question. So we should define C and S, before considering C+iS.

So let $C = \cos\theta + \frac{1}{3}\cos 3\theta + \frac{1}{9}\cos 5\theta + \ldots$

$S = \sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{9}\sin 5\theta + \ldots$

So considering $C + iS$,

$C + iS = \cos\theta + i\sin\theta$

$+ \frac{1}{3}\cos 3\theta + \frac{1}{3}i\sin 3\theta$

$+ \frac{1}{9}\cos 5\theta + \frac{1}{9}i\cos 5\theta$

$+ \ldots$

$= z + \frac{1}{3}z^3 + \frac{1}{9}z^5$

This forms a geometric series, $a = z$, $r = \frac{1}{3}z^2$, summed to infinity.

(answer space continued on next page)

14(b) **(continued)**

infinite geometric sum formula.

so $C + iS = \frac{a}{1-r} = \frac{z}{1-\frac{1}{3}z^2}$

$= \frac{3z}{3-z^2}$

$= \frac{3e^{i\theta}}{3-e^{2i\theta}} \times \frac{3-e^{-2i\theta}}{3-e^{-2i\theta}}$

multiplying by complex conjugate

$= \frac{3e^{i\theta}(3-e^{-2i\theta})}{(3-e^{2i\theta})(3-e^{-2i\theta})}$

$= \frac{9e^{i\theta} - 3e^{-i\theta}}{10-6\cos 2\theta}$

using result from a)

$= \frac{9(\cos\theta + i\sin\theta) - 3(\cos(-\theta) + i\sin(-\theta))}{10-6\cos 2\theta}$

using $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$

$= \frac{9\cos\theta + 9i\sin\theta - 3\cos\theta + 3i\sin\theta}{10-6\cos 2\theta}$

$= \frac{6\cos\theta}{10-6\cos 2\theta} + i\,\frac{12\sin\theta}{10-6\cos 2\theta}$

$= \frac{3\cos\theta}{5-3\cos 2\theta} + i\,\frac{6\sin\theta}{5-3\cos\theta}$

hence $C = \frac{3\cos\theta}{5-3\cos 2\theta}$ and $S = \frac{6\sin\theta}{5-3\cos 2\theta}$

so $\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{9}\sin 5\theta + \ldots = \frac{6\sin\theta}{5-3\cos 2\theta}$

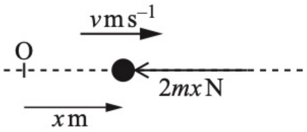
as required

15 In an oscillating system, a particle of mass m kg moves in a horizontal line. Its displacement from its equilibrium position O at time t seconds is x metres, its velocity is $v\,\mathrm{m\,s^{-1}}$, and it is acted on by a force $2mx$ newtons acting towards O as shown in the diagram.

G

Initially, the particle is projected away from O with speed $1\,\mathrm{m\,s^{-1}}$ from a point 2 m from O in the positive direction.

(a) (i) Show that the motion is modelled by the differential equation $\frac{d^2x}{dt^2} + 2x = 0$. **[1]**



The resultant force on the particle, F, is

$F = -2mx$

By Newton's 2^{nd} Law, $F = ma$

using $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

$-2\cancel{m}x = \cancel{m}a$

$\frac{d^2x}{dt^2} = -2x \Rightarrow \frac{d^2x}{dt^2} + 2x = 0$ as required

G

(ii) State the type of motion. **[1]**

Since $a \propto -x$, the particle is performing simple harmonic motion.

G

(iii) Write down the period of the motion. **[1]**

For SHM, $\frac{d^2x}{dt^2} + \omega^2 x = 0$. So $\omega^2 = 2 \Rightarrow \omega = \sqrt{2}$.

Also recall time period, T, is given by $T = \frac{2\pi}{\omega}$.

So $T = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\,\pi = 4.4428\ldots$

$= 4.44$ seconds

A

(iv) Find x in terms of t. **[4]**

Recall that the general solution for a SHM DE is

$x = A\cos\omega t + B\sin\omega t$.

So $x = A\cos(t\sqrt{2}) + B\sin(t\sqrt{2})$

At $t = 0$, $x = 2$ and $v = \frac{dx}{dt} = 1$

$A\cos(0) + B\sin(0) = 2$

$A(1) + B(0) = 2 \Rightarrow A = 2$

$x = 2\cos(t\sqrt{2}) + B\sin(t\sqrt{2})$

differentiate

$\frac{dx}{dt} = -2\sqrt{2}\sin(t\sqrt{2}) + B\sqrt{2}\cos(t\sqrt{2})$

$1 = -2\sqrt{2}\sin(0) + B\sqrt{2}\cos(0) \Rightarrow B = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

hence $x = 2\cos(t\sqrt{2}) + \frac{\sqrt{2}}{2}\sin(t\sqrt{2})$

A

(v) Find the amplitude of the motion. **[2]**

Recall that, if $x = A\cos\omega t + B\sin\omega t$, the amplitude of motion is given by $\sqrt{A^2+B^2}$.

hence amplitude $= \sqrt{2^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} = 2.1213\ldots$

$= 2.12\text{m}$ (3sf)

G

(b) The motion is now damped by a force $2mv$ newtons.

(i) Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0$. **[1]**

The resultant force, R, on the particle is now

$R = -2mx - 2mv$

So by Newton's 2nd Law, $-2\cancel{m}x - 2\cancel{m}v = \cancel{m}a$

$\left(a = \frac{d^2x}{dt^2},\ v = \frac{dx}{dt}\right)$ $a + 2v + 2x = 0 \Rightarrow \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0$ as required

A

(ii) State, giving a reason, whether the system is under-damped, critically damped or over-damped. **[1]**

We can consider the auxilliary equation for our DE,

$\lambda^2 + 2\lambda + 2 = 0.$

The discriminant $= (2)^2 - 4(1)(2) = -4$

Recall that if discriminant is less than 0, the particle is underdamped. Hence as discriminant < 0, it is under-damped.

A

(iii) Determine the general solution of this differential equation. **[3]**

We can use our auxilliary equation here.

$\lambda^2 + 2\lambda + 2 = 0$

$\lambda = \frac{-2 \pm \sqrt{(2)^2 - 4\times1\times1}}{2\times1} = -1 \pm i.$

If we obtain roots $\lambda = a \pm bi$ from our auxilliary equation, $x = e^{at}(A\cos bt + B\sin bt)$.

So our general solution is $x = e^{-t}(A\cos t + B\sin t)$

(c) Finally, a variable force $2m\cos 2t$ newtons is added, so that the motion is now modelled by the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 2\cos 2t.$$

R

(i) Find x in terms of t. [7]

From b), we now have a general solution

$$x = e^{-t}(A\cos t + B\sin t)$$

So our solution is

$$x = e^{-t}(A\cos t + B\sin t) + p(t)$$

We can find $p(t)$.

$$p(t) = P\cos 2t + Q\sin 2t$$
$$p'(t) = -2P\sin 2t + 2Q\cos 2t$$
$$p''(t) = -4P\cos 2t - 4Q\sin 2t$$

Subbing into the DE,

$$-4P\cos 2t - 4Q\sin 2t + 2(-2P\sin 2t + 2Q\cos 2t)$$
$$+ 2(P\cos 2t + Q\sin 2t) = 2\cos 2t$$
$$-4P\cos 2t - 4Q\sin 2t - 4P\sin 2t + 4Q\cos 2t$$
$$+ 2P\cos 2t + 2Q\sin 2t = 2\cos 2t$$

$\cos 2t$: $-4P\cos 2t + 4Q\cos 2t + 2P\cos 2t = 2\cos 2t$

$$-2P + 4Q = 2$$
$$-P + 2Q = 1 \quad (A)$$

$\sin 2t$: $-4Q\sin 2t - 4P\sin 2t + 2Q\sin 2t = 0$

$$-4P - 2Q = 0$$
$$2P + Q = 0 \quad (B)$$

Solving (A) and (B) simultaneously gives

$$P = -\frac{1}{5} \text{ and } Q = \frac{2}{5}$$

Hence $x = e^{-t}(P\cos t + Q\sin t) - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$

(answer space continued on next page)

15(c)(i) **(continued)**

In the long term, the particle is seen to perform simple harmonic motion with a period of just over 3 seconds.

A

(ii) Verify that this behaviour is consistent with the answer to part **(c)(i)**. **[2]**

As $t \to \infty$, $e^{-t} \to 0$.

So $x \to -\frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$

We can rewrite x as $t \to \infty$ in harmonic form,

$x \to A\sin(2t + \alpha)$

so $\omega = 2$.

Hence time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi = 3.141.....$

This time period is close to 3 seconds as expected.

ADDITIONAL ANSWER SPACE

If additional space is required, you should use the following lined page(s). The question number(s) must be clearly shown in the margin(s).